sistance. Combined with this fact is the fact that the radial velocity supports the swirl by convection of angular momentum. Thus, the radial velocity always distributes itself axially in a way that tends to make the tangential velocity two-dimensional as far as is possible.

In strongly rotating flows, the effect of the rotation on the stream function is dependent on the gradient of the basic circulation. The larger the circulation gradient, the steeper the adjustments in the shear layers.

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Couette Flow of a Radiating and Conducting Gas

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This paper considers the Couette flow of an absorbing, emitting, and conducting gas. The combined radiation and conduction problem is treated by 1) kernel substitution and 2) radiation slip methods. Results are presented for the heat flux and temperature distributions for a gray gas. In general, there is good agreement between the kernel substitution method and the radiation slip plus conduction method for the determination of the heat flux. In the absence of conduction, the temperature distribution obtained from the kernel substitution method gives slightly better agreement with numerical results than the results obtained from radiation slip methods.

Nomenclature

k	=	absorption coefficient
T^*	=	temperature
$T_0{}^*$	=	reference temperature, taken to be $T_W{}^*$
T	=	dimensionless temperature, $T^*/T_0{}^*$
f	=	dimensionless freestream temperature
y	=	distance from left wall
I	=	radiation intensity
q^*	=	total heat flux
q	=	dimensionless heat flux, $q^*/\sigma T_0{}^{*4}$
$E_n(t)$	=	exponential integral $= \int_0^1 \mu^{n-2} e^{-t/\mu} d\mu$
λ	=	thermal conductivity
τ^*	=	optical depth, $\int_0^1 k dy$
τ	=	$3\tau^*/2$
σ	=	Stefan-Boltzmann constant
μ	=	viscosity
u	=	velocity
U	=	velocity of upper wall
ϕ	=	$\mu U^2/2y_W\sigma T_0{}^{*4}$
ϵ	=	$3\lambda k/4\sigma T_0{}^{*3}\tau_W{}^2$
δ	=	$\epsilon\tau_W{}^2$
ξ	=	y/y_W
$\tilde{\xi}$	=	$\xi/\epsilon^{1/2}$

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Superscripts

(0) = constants evaluated as $\delta \to 0$
$+$ = constants evaluated as $\tau_W \to \infty$
0 = blackbody radiation

Subscripts

R = radiation
1 = lower wall
w = upper wall
a = $\begin{cases} 1 \text{ if lower wall is being considered} \\ w \text{ if upper wall is being considered} \end{cases}$
ν = per unit frequency

Introduction

THE interaction of radiation with other modes of heat transfer is an important consideration in many high-temperature problems. The complexity of the radiative transfer relations, in addition to the critical importance of the variation of properties, excludes the posibility of exact analytical solution for real problems. Several approximation ideas have been suggested, and this study will provide a useful comparison of results for the problem of Couette flow of a radiating and conducting gas with viscous dissipation.

Couette Flow Equations

We consider the flow of an absorbing, emitting, and conducting gas between two infinite parallel flat walls. One wall moves with constant velocity U; the other is at rest. The walls are kept at constant temperatures T_1 and T_W with emissivities ϵ_1 and ϵ_W.

The equations for the conservation of momentum and energy for Couette flow are given by[1]

$$d/dy\ [\mu(du/dy)] = 0 \tag{1}$$

$$\frac{d}{dy}\left(-\lambda\frac{dT^*}{dy} + q_R^*\right) - \mu\left(\frac{du}{dy}\right)^2 = 0 \tag{2}$$

where q_R^*, the radiative heat flux, is given by[2, 3]

$$q_R^* = 2\pi\int_0^\infty\int_0^{\tau_\nu^*} I_\nu^0\, E_2(\tau_\nu^* - t)\, dtd\nu - 2\pi\int_0^\infty\int_{\tau_\nu^*}^{\tau_{w^*}} I_\nu^0 E_2(t - \tau_\nu^*)\, dtd\nu + 2\int_0^\infty q_{\nu_1}^* E_3\,(\tau_\nu^*)\, d\nu - 2\int_0^\infty q_{\nu_w}^* E_3\,(\tau_{\nu_w}^* - \tau_\nu^*)\, d\nu \tag{3}$$

I_ν^0 is the blackbody spectral intensity, and $q_{\nu_1}^*$ and $q_{\nu_w}^*$ are the radiative spectral fluxes leaving the walls. The other quantities are defined in the Nomenclature. Equation (2) is analogous to the energy equation resulting from the electric arc heating of a gas under conditions of laminar flow, i.e., in the "Poiseuille Plasma Experiment."[4] The rate of dissipation of mechanical energy into heat is then replaced by the electrical energy dissipation.

Substitute Kernel

Solution of the preceding equations requires numerical methods. To correct this difficulty, the exact kernel $E_2(t)$ is approximated by the exponential function $\frac{3}{4}e^{-3t/2}$, which has the same area and the same first moment as the exact kernel. Lick[3] applied this approximation to a combined conduction and radiation problem and obtained very good agreement with the results obtained from the numerical solution of the original equation.[5] In this study we will only consider a gas with constant properties. It is important to note, however, that the kernel substitution method can be extended to variable gas properties.[6] The importance of considering varying properties cannot be over emphasized.[7, 4]

Making the kernel substitution and taking successive differentiations of Eq. (2) yields the differential equation

$$\epsilon\frac{d^2T}{d\xi^2} - \epsilon\tau_W^2 T - T^4 = -\alpha - \beta\tau_W\xi + \frac{\phi}{2}\tau_W\xi^2 \tag{4}$$

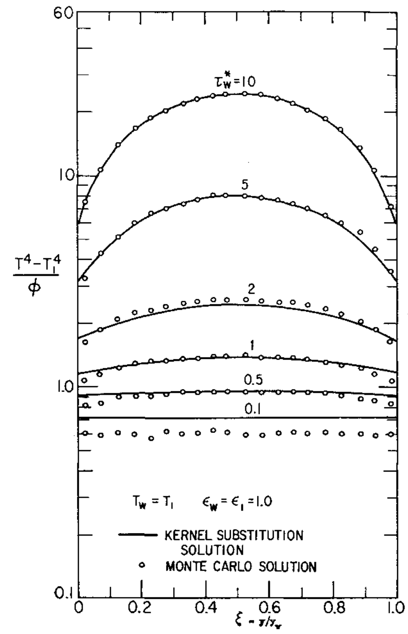

Fig. 1 Temperature distribution without conduction.

where

$$\alpha = \frac{1}{2+\tau_W}\left\{q^-(\tau_W) + q^+(0)[1+\tau_W] + \frac{\phi}{2\tau_W}(2+\tau_W)^2 + \delta\left[T_W + T_1(1+\tau_W) + \left.\frac{dT}{d\tau}\right|_W \left.\frac{-dT}{d\tau}\right|_1 (1+\tau_W)\right]\right\} \tag{5}$$

$$\beta = \frac{1}{2+\tau_W}\left\{q^-(\tau_W) - q^+(0) + \frac{\phi}{2}(2+\tau_W) + \delta\left[T_W - T_1 + \left.\frac{dT}{d\tau}\right|_W + \left.\frac{dT}{d\tau}\right|_1\right]\right\} \tag{6}$$

$$\phi = \varphi U^2/2y_W\sigma T_0^{*4} \tag{7}$$

$$\delta = \epsilon\tau_W^2 = \tfrac{3}{4}(\lambda k/\sigma T_0^{*3}) \tag{8}$$

The heat flux is given by

$$-q = 2\beta - 2\phi\xi \tag{9}$$

Boundary-Layer Analysis

When radiation heat transfer is dominant ($\epsilon \ll 1$) the effect of conduction is restricted to a boundary layer close to each wall. For $\delta \ll 1$, the freestream solution ($\epsilon = 0$) is

$$T^4 = f^4 = \alpha^{(0)} + \beta^{(0)}\tau_W\xi - (\phi/2)\tau_W\xi^2 \tag{10}$$

where $\alpha^{(0)}$ and $\beta^{(0)}$ are the values of α and β obtained for $\delta = 0$. Equation (10) can be viewed as the temperature distribution for the Couette flow of a radiating gas with no conduction. The result for equal wall temperatures and emissivities is given by

$$\frac{(T^4 - T_1^4)}{\phi/\tau_W^*} = \tau_W^*\left(\frac{1}{\epsilon} - \frac{1}{2}\right) + \frac{2}{3} + \frac{3}{4}\tau_W^{*2}(\xi - \xi^2) \tag{11}$$

and is plotted in Figs. 1 and 2. Included for comparison are the curves for the Monte Carlo solution of Howell and Perlmutter[8] for the equivalent radiation plus heat source problem. The agreement is seen to be very good over the entire range of emissivities and optical depths.

The boundary-layer equation is obtained by stretching the length variable such that the most highly differentiated term is of the same order of magnitude as the largest terms in the

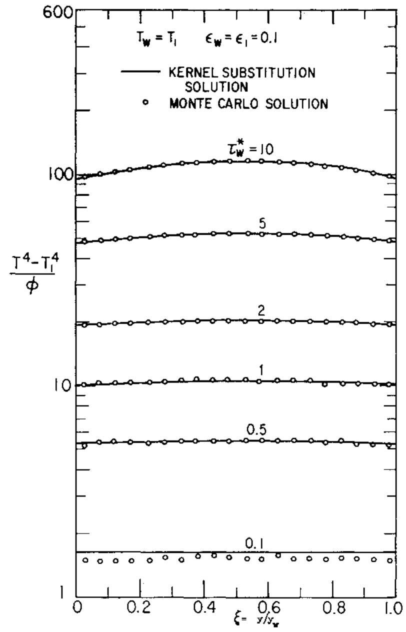

Fig. 2 Temperature distribution without conduction.

equation. This is obtained with the transformation $\bar{\xi} = \xi/\epsilon^{1/2}$, and Eq. (4) becomes

$$\frac{d^2T}{d\bar{\xi}^2} - T^4 = -\alpha^{(0)} - \beta^{(0)}\tau_W\xi_a + \frac{\phi}{2}\tau_W\xi_a^2 = -f_a^4 \quad (12)$$

where the subscript a is 1 for the boundary layer at the lower wall and W for the boundary layer at the upper wall. Integrating yields

$$\delta\left(\frac{dT}{d\tau}\right)_a = (2\delta)^{1/2}\left[\frac{1}{5}(T_a^5 - f_a^5) - f_a^4(T_a - f_a)\right]^{1/2} \quad (13)$$

A numerical integration of Eq. (13) yields the temperature distribution.

Power Series Expansion

When conduction is dominant $\epsilon \gg 1$ we approximate the solution by the power series

$$T = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 \quad (14)$$

The coefficients a_n are determined by substituting Eq. (14) into Eq. (4). In the limit as $\epsilon \to \infty$, Eq. (14) reduces to the pure conduction solution.

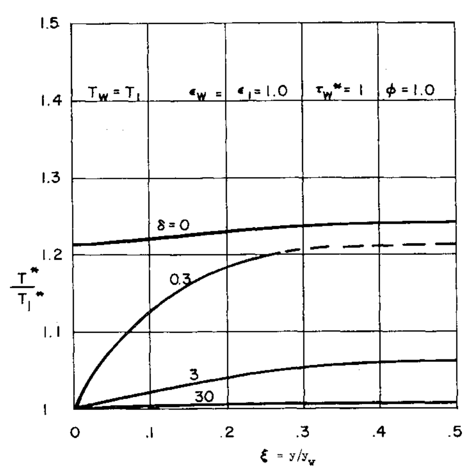

Fig. 3 Temperature distribution for radiating and conducting Couette flow.

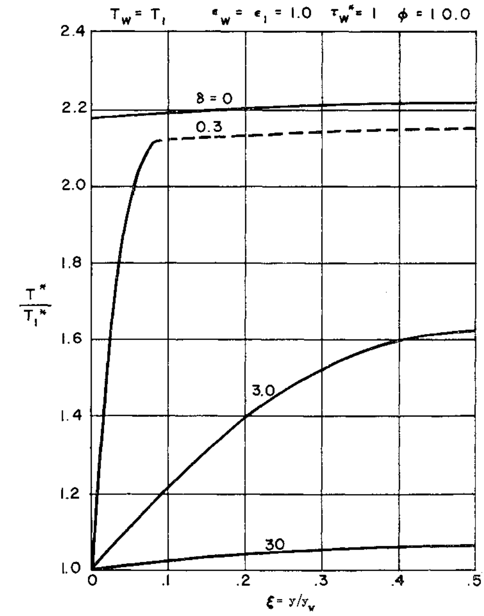

Fig. 4 Temperature distribution for radiating and conducting Couette flow.

Diffusion Approximation

When the medium has a large optical depth ($\tau_W \gg 1$) Eq. (4) may be approximated by

$$\delta T + T^4 = \alpha^+ + \beta^+\tau_W\xi - \frac{\phi}{2}\tau_W\xi^2 \quad (15)$$

where

$$\alpha^+ = \delta T_1 + T_1^4 \quad (16)$$

$$\beta^+ = \frac{\delta(T_W - T_1)}{\tau_W} + \frac{T_W^4 - T_1^4}{\tau_W} + \frac{\phi}{2} \quad (17)$$

$$-q = 2\delta\frac{(T_W - T_1)}{\tau_W} + \frac{2(T_W^4 - T_1^4)}{\tau_W} + \phi(1 - 2\xi) \quad (18)$$

These results may be directly obtained by approximating the radiative heat flux by the Rosseland diffusion relation[4] (see Viskanta and Grosh[10]).

Results

The temperature distribution has been plotted in Figs. 3 and 4 for the case of equal wall temperatures for $\tau_W^* = 1$ with $\epsilon_1 = \epsilon_W = 1$ for $\phi = 1$ and for $\phi = 10$. When conduction dominates ($\delta \gg 1$) the temperature distribution varies slowly with the distance across the channel. When radiation dominates ($\delta \ll 1$) the effects of conduction are

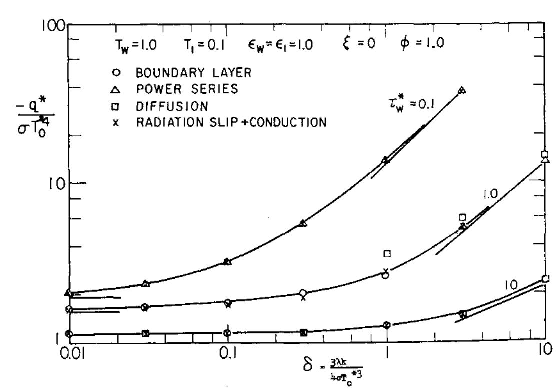

Fig. 5 Heat flux for radiating and conducting Couette flow.

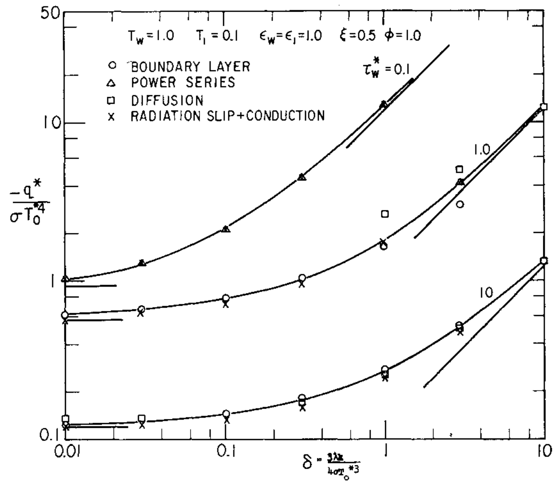

Fig. 6 Heat flux for radiating and conducting Couette flow.

primarily restricted to a region close to the wall. In the absence of conduction there is a temperature discontinuity at the wall (also see Figs. 1 and 2). For equal wall temperatures, the total heat flux may be determined directly from Eq. (9) to give

$$-q = \phi(1 - 2\xi) \tag{19}$$

All the previous limiting cases correctly reduce to this result. The heat flux is plotted in Figs. 5–10 for unequal wall temperatures $T_W = 1$ and $T_1 = 0.1$, for values of ξ equal to 0, 0.5, and 1, for $\phi = 1$, and for $\phi = 10$. For the larger values of the optical depth τ_W^* the effects of dissipation dominate in the important regions beyond the center of the channel. This can be seen in Figs. 7 and 10, which show that the net heat flux is directed towards the hot wall for the larger values of τ_W^*. The effect is, of course, more pronounced for increasing dissipation.

Radiation Slip

The problem of calculating the radiative heat transfer may also be determined by extending the range of validity of the Rosseland diffusion approximation. This may be accomplished by introducing a temperature jump boundary condition at the gas solid surface.[11–15] The result for the radiative heat transfer between parallel plates is given by

$$-q = \frac{\sigma(T_W^4 - T_1^4)}{1 + (3\tau_W^*/4)} \tag{20}$$

Equation (20) is in good agreement with the numerical results of the original equation over the entire range of optical depths. It should be noted that there is little theoretical

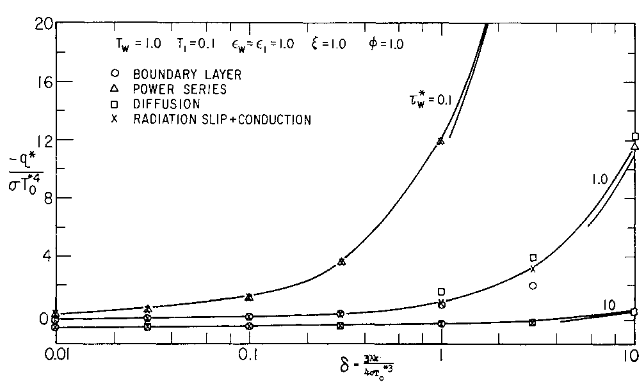

Fig. 7 Heat flux for radiating and conducting Couette flow.

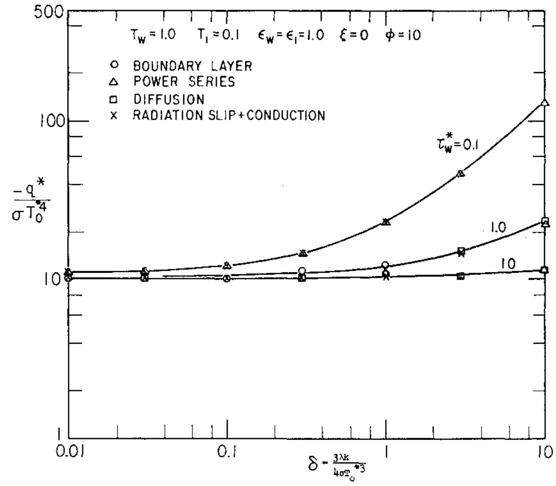

Fig. 8 Heat flux for radiating and conducting Couette flow.

justification in extending Eq. (20) to very small values of τ_W^*. However, the analogous rarefied gas problem also gives good agreement over the entire range of mean free paths. A more detailed discussion is available.[13, 16]

For the problem of the Couette flow of a radiating gas in the absence of conduction, it is possible to make direct comparisons with the kernel substitution method. The temperature distribution obtained from the first-order temperature jump condition and matching procedure of Probstein[13] differs from Eq. (11) by omitting the constant term $\frac{2}{3}$. A comparison with the numerical results of Fig. 1 reveals poor agreement for small optical depths. Using second-order energy jump conditions,[12, 8] the resulting temperature distribution differs from Eq. (11) by having a different constant, namely, $\frac{3}{4}$. This result is in better agreement with the numerical results of Fig. 1 but is still slightly inferior to the kernel substitution method.

Probstein[13] has included the effects of conduction by calculating the net heat-transfer rate as two separate heat fluxes in parallel. This cannot be theoretically justified for all optical depths but appears to work rather well. This approximate approach may also be extended to the problem of Couette flow of a radiating and conducting gas. The total heat flux is then given by

$$-q = \frac{(T_W^4 - T_1^4)}{1 + (3\tau_W^*)/4} + \delta\,\frac{(T_W - T_1)}{\frac{3}{4}\tau_W^*} + \phi(1 - 2\xi) \tag{21}$$

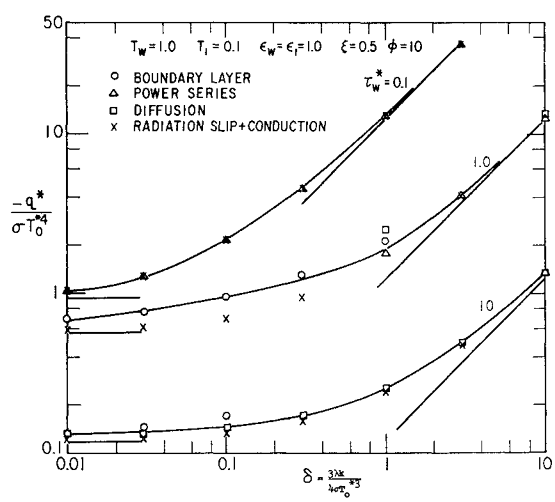

Fig. 9 Heat flux for radiating and conducting Couette flow.

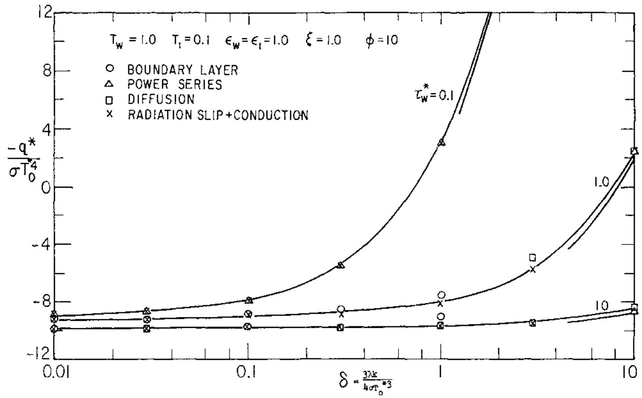

Fig. 10 Heat flux for radiating and conducting Couette flow.

For the special case of equal wall temperatures, Eq. (21) reduces to the correct result given by Eq. (19). For unequal wall temperature, $T_W = 1$ and $T_1 = 0.1$, the heat flux, as determined by Eq. (21), is plotted in Figs. 5–10. Good agreement is obtained with the results obtained from the kernel substitution method.

Summary

The problem of the Couette flow of a radiating and conducting gas has been studied. The results for the heat flux and temperature distributions are presented in terms of the basic parameters; the optical depth τ_W^*, the conduction to radiation parameter δ, the dissipation function 2ϕ, and the nondimensional position ξ. In general, there is good agreement between the kernel substitution method and the radiation slip plus conduction method for the determination of the heat flux. A comparison with numerical results shows that the kernel substitution method is slightly superior to radiation slip methods for the determination of the temperature distribution. We note that the kernel substitution method can be extended to gases having spectral and temperature dependent properties.[3,6]

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